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BLENDING INTERPOLATION SCHEMES ON TRIANGLES WITH ERROR BOUNDS.(U)

APR 77 K BOEHMER, G COMAN DAAG29-75-C-0024

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MRC Technical Summary Report #1744

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April 1977

(Received January 4, 1977)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Volkswagen Foundation
Germany

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

We give rational approximation schemes interpolating the values of a given function along two edges of a triangle and the normal derivative along the third edge. In addition, we give error bounds for our schemes. For the uniform norm, these bounds are best possible.

AMS (MOS) Subject Classifications: 41A05, 41A15, 41A63, 65D05

Key Words: Surface interpolation, Blending interpolation on triangles, Interpolation error

Work Unit Number 6 (Spline Functions and Approximation Theory)

Sponsored in part by the United States Army under Contract No. DAAG29-75-C-0024, in part by the Volkswagen Stiftung, Deutschland under Contract No. 112256 and a Fulbright Fellowship.

BLENDING INTERPOLATION SCHEMES ON TRIANGLES WITH ERROR BOUNDS

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0. Introduction:

A few years ago Barnhill-Birkhoff-Gordon [1] constructed rational functions, interpolating the values of a given function on the boundary of a certain normalized triangle T. In the same paper interpolation schemes are given, realizing the values of a given function and its normal derivatives on the boundary of T. These schemes are affinely invariant iff the normal derivatives are transformed into the corresponding directional derivatives.

Here we give rational approximation schemes interpolating the values of a given function along two edges of a triangle T and the normal derivative along the third edge. In addition we study the remainders of our schemes and give better error bounds than in [6]. For the L_∞-norm the error bounds are best possible.

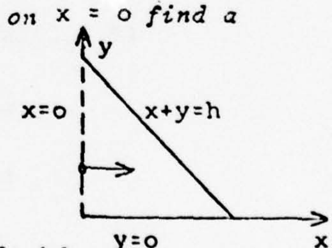
As triangle T we choose the standard triangle $T_h=\{(x,y)\in \mathbb{R}^2 \mid x \geqq o,\ y \geqq o,\ x + y \leqq h\}$ with vertices (o,o), (o,h) and (h,o). Our schemes are affinely invariant in the same sense as discussed above.

1. First Interpolationscheme:

In contrary to the following problems we treat Problem 1 fairly extensively.

Problem 1: *For* $f : T_h \to \mathbb{R}$ *with* $f^{(1,o)}$ *existing on* $x = o$ *find a blending function* $G_1 f$ *such that*

$$(1)\quad \begin{cases} G_1 f|_{y=o} = f|_{y=o} \text{ for } x \in [o,h]\,, \\ G_1 f|_{x+y=h} = f|_{x+y=h} \text{ for } x \in [o,h]\,, \\ (G_1 f)^{(1,o)}|_{x=o} = f^{(1,o)}|_{x=o} \text{ for } y \in [o,h]\,. \end{cases}$$



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As in many similar cases we use the Boolean sum of two operators $\mathcal{A}, \mathcal{L}$, namely

$$\mathcal{A} \oplus \mathcal{L} := \mathcal{A} + \mathcal{L} - \mathcal{A}\mathcal{L} \quad , \qquad \begin{array}{l} \mathcal{A} : J \to J \\ \mathcal{L} : J \to J. \end{array}$$

Here we take $\mathcal{A} = P_1$, $\mathcal{L} = P_2$ with

$$P_1 f|_{x+y=h} = f|_{x+y=h} = P_2 f|_{x+y=h} \quad ,$$

$$(P_1 f)^{(1,o)}|_{x=o} = f^{(1,o)}|_{x=o} \quad ,$$

$$P_2 f|_{y=o} = f|_{y=o} \quad .$$

Immediately one finds (compare the technique used in [1])

$$(P_1 f)(x,y) = f(h-y,y) + (x+y-h) f^{(1,o)}(o,y)$$

$$(P_2 f)(x,y) = \frac{h-x-y}{h-x} f(x,o) + \frac{y}{h-x} f(x,h-x)$$

and therefore

$$G_1 f = P_1 f \oplus P_2 f.$$

We find for $G_1 f$ with

$$U_1 := P_1 f, \; U_2 := P_2 f$$

the formula

$$\begin{aligned}(P_1 P_2 f)(x,y) &= (P_1 U_2)(x,y) \\ &= f(h-y,y) + (x+y-h) \Big[\frac{y}{h^2}(f(o,h)-f(o,o)) \\ &\quad + \frac{h-y}{h} f^{(1,o)}(o,o) + \frac{y}{h} (f^{(1,o)}(o,h) - f^{(o,1)}(o,h))\Big]\end{aligned}$$

and so finally

$$(2) \left\{ \begin{aligned} (G_1 f)(x,y) = &\frac{h-x-y}{h-x} f(x,o) + \frac{y}{h-x} f(x,h-x) + (x+y-h) f^{(1,o)}(o,y) \\ &+ (h-x-y) \Big[\frac{y}{h^2}(f(o,h) - f(o,o)) + \frac{h-y}{h} f^{(1,o)}(o,o) \\ &\qquad + \frac{y}{h}(f^{(1,o)}(o,h) - f^{(o,1)}(o,h))\Big] \quad . \end{aligned} \right.$$

Now it is straightforward to check that G_1 safisfies (1) and it is equally simple to prove the rest of the following.

Theorem 1: *Let* $f : T_h \to \mathbb{R}$ *be given.*

(a) *If* $f^{(1,o)}(o,y)$ *for* $y \in [o,h]$ *and* $f^{(o,1)}(o,h)$ *exist, then* $G_1 f$ *in* (2) *satisfies* (1). *Moreover, if* $f(\cdot,o)$, $f(\cdot,h-\cdot)$ *and* $f^{(1,o)}(o,\cdot) \in C^j[o,h]$ *then* $G_1 f \in C(T_h)$ *for* $j = o$ *and* $G_1 f \in C^j(T_h \setminus \{(h,o)\})$ *for* $j > o$.

(b) G_1 *reproduces bivariate polynomials of total degree at most* 2.

(c) *If* $f(\cdot,o)$, $f(\cdot,h-\cdot)$ *and* $f^{(1,o)}(o,\cdot) \in C^2[o,h]$, *then* $G_1 f$ *is a solution of*

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)^2 \phi = o \text{ in } T_h \setminus \{(h,o)\} .$$

(d) *If* $f(\cdot,o)$, $f(o,\cdot)$, $f^{(1,o)}(o,\cdot) \in L_\infty[o,h)$ *then* G_1 *satisfies the following kind of a maximum principle*

$$\|G_1 f\|_{L_\infty(T_h)} \leq \|f(\cdot,o)\|_{L_\infty[o,h]} + \|f(o,\cdot)\|_{L_\infty[o,h]}$$

$$+ h\left\| f^{(1,o)}(o,\cdot) + \frac{h-\cdot}{h} f^{(1,o)}(o,o)\right\|_{L_\infty[o,h]}$$

$$+ \frac{1}{4}\left| f(o,h)-f(o,o) + h(f^{(1,o)}(o,h) - f^{(o,o)}(o,h))\right| .$$

3. Remainder for the First Interpolation Scheme:

To estimate the remainder $R_1 f$ in

(3) $\qquad R_1 f := f - G_1 f \quad$ for $G_1 f$ in (2)

we use a "Sard-Kernel-Theorem" [16] on triangles due to Barnhill-Mansfield [2]: Let F be a linear functional of the form

(4) $$F(g) = \sum_{\substack{i<p\\ j<q}} \iint_T g^{(i,j)}(x,y)\,d\mu_{ij}(x,y) +$$

$$+ \sum_{\substack{i+j<m\\ i\geq p}} \int_{I_1} g^{(i,j)}(x,b)\,d\mu_{ij}(x) + \sum_{\substack{i+j<m\\ j\geq q}} \int_{I_2} g^{(i,j)}(a,y)\,d\mu_{ij}(y),$$

where the functions μ_{ij} are of bounded variation, T is a given triangle, $(a,b) \in T$ is such that the rectangle with opposite vertices (a,b) and (x,y) and with edges parallel to the coordinate axes is contained in T for all $(x,y) \in T$, and I_1, I_2 are the intersections of the line $y = b$ respectively $x = a$ with T. If F satisfies

$$F(p) = 0,\ p \in \mathbb{P}_{m-1} = \{\text{polynomials of total degree} \leq m-1\}\ ,$$

then we have

$$(5)\quad F(g) = \sum_{j<q} \int_{I_1} K^{m-j,j}(x,y;t)g^{(m-j,j)}(t,b)dt$$

$$+ \sum_{i<p} \int_{I_2} K^{i,m-i}(x,y;\tau)g^{(i,m-i)}(a,\tau)d\tau$$

$$+ \iint_T K^{p,q}(x,y;t,\tau)g^{(p,q)}(t,\tau)dt\,d\tau,\ g \in B^1_{p,q}$$

where

$$(6)\begin{cases} K^{m-j,j}(x,y;t) = F_{(x,y)}\left[\dfrac{(x-t)^{m-j-1}}{(m-j-1)!}\,\psi(a,t,x)\dfrac{(y-b)^j}{j!}\right],\ j < q \\[2ex] K^{i,m-i}(x,y;\tau) = F_{(x,y)}\left[\dfrac{(x-a)^i}{i!}\,\dfrac{(y-\tau)^{m-i-1}}{(m-i-1)!}\psi(b,\tau,y)\right],\ i < p \\[2ex] K^{p,q}(x,y;t,\tau) = F_{(x,y)}\left[\dfrac{(x-t)^{p-1}}{(p-1)!}\psi(a,t,x)\dfrac{(y-\tau)^{q-1}}{(q-1)!}\psi(b,\tau,y)\right] \end{cases}$$

with

$$\psi(a,t,x) = \begin{cases} 1 \text{ if } a \leq t < x \\ -1 \text{ if } x \leq t < a \\ 0 \text{ otherwise,} \end{cases}$$

and $B^{1,m}_{p,q} = B^{1,m}_{p,q}(a;b)$, i.e. the class of functions $g : T \to R$ with the properties

$$(7)\begin{cases} g^{(i,j)} \in C(T),\ i < p,\ j < q, \\ g^{(m-j-1,j)}(\cdot,b) \in C\,(I_1) \quad ,\ g^{(m-j,j)}(\cdot,b) \in L_1(I_1),\ 0 \leq j < q \\ g^{(i,m-i-1)}(a,\cdot) \in C\,(I_2) \quad ,\ g^{(i,m-i)}(a,\cdot) \in L_1(I_2),\ 0 \leq i < p \\ g^{(p,q)} \in L_1(T). \end{cases}$$

We, too, will use the class $B^{r,m}_{p,q} = B^{r,m}_{p,q}(a;b)$ of functions, $r \geqq 1$, whith slightly modified properties (7) namely

$$g^{(m-j,j)}(\cdot,b) \in L_r(I_1),\ 0 \leqq j < q$$

$$g^{(i,m-i)}(a,\cdot) \in L_r(I_2),\ 0 \leqq i < p$$

$$g^{(p,q)} \in L_r(T).$$

Now we are able to prove the following

<u>Theorem 2:</u> *If* $f \in B^{\infty,3}_{1,2}$ (o;o) *then*

$$(8)\quad \|R_1 f\|_{L_\infty(T_h)} \leqq \frac{h^3}{27}\|f^{(2,1)}\|_{L_\infty(T_h)} + \frac{2h^3}{81}\|f^{(o,3)}(o,\cdot)\|_{L_\infty[o,h]}$$

$$+ \frac{2h^3}{27}\|f^{(1,2)}(o,\cdot)\|_{L_\infty[o,h]},$$

where the constants are best possible. If $f \in B^{2,3}_{1,2}(o;o)$ *then*

$$(9)\quad \|R_1 f\|_{L_2(T_h)} \leqq \frac{h^3}{6\cdot\sqrt{3o}}\|f^{(2,1)}\|_{L_2(T_h)} +$$

$$+ h^{7/2}\frac{\sqrt{397/2}}{126o}\|f^{(o,3)}(o,\cdot)\|_{L_2[o,h]} + h^{7/2}\frac{\sqrt{19/14}}{2o}\|f^{(1,2)}(o,\cdot)\|_{L_2[o,h]}.$$

<u>Remark 1:</u> Best possible, but unusual, estimations for the L_2-norm are valid if $f \in B^{\infty,3}_{1,2}(o;o)$:

$$\|R_1 f\|_{L_2(T_h)} \leqq \frac{h^4}{2o\sqrt{21}}\|f^{(2,1)}\|_{L_\infty(T_h)} + \frac{h^4}{36\sqrt{7}}\|f^{(o,3)}(o,\cdot)\|_{L_\infty[o,h]} +$$

$$+ \frac{h^4\sqrt{19/1o5}}{4}\|f^{(1,2)}(o,\cdot)\|_{L_\infty[o,h]}.$$

This is proved by (13), (17) via $\iint (\overset{2,1}{g}(x,y))^2 dxdy$ with $g^{2,1}$ from (15) and similar arguments for the other indices.

Proof: Since, by (2) and (3), $R_1 f$ is of the form (4) with $m = 3$, $p = 2$, $q = 1$, $a=b=o$ we find, using (5),

$$
(10)\quad \begin{cases}
(R_1 f)(x,y) = \int\limits_o^h K_h^{3,o}(x,y;t) f^{(3,o)}(t,o)\,dt \\
+ \int\limits_o^h K_h^{o,3}(x,y;\tau) f^{(o,3)}(o,\tau)\,d\tau + \int\limits_o^h K_h^{1,2}(x,y;\tau) f^{(1,2)}(o,\tau)\,d\tau \\
+ \iint\limits_{T_n} K_h^{2,1}(x,y;t,\tau) f^{(2,1)}(t,\tau)\,dt\,d\tau .
\end{cases}
$$

Now (6) implies

$$
(11)\quad \begin{cases}
K_h^{3,o}(x,y;t) = R_1\left[\frac{(x-t)_+^2}{2}\right] \\
= \frac{(x-t)_+^2}{2} - \frac{h-x-y}{h-x}\frac{(x-t)_+^2}{2} - \frac{y}{h-x}\frac{(x-t)_+^2}{2} \equiv o \\
K_h^{o,3}(x,y;\tau) = R_1\left[\frac{(y-\tau)_+^2}{2}\right] = \frac{(y-\tau)_+^2}{2} - \{\frac{y}{h-x}\frac{(h-x-\tau)_+^2}{2} \\
\qquad - \frac{(h-x-y)y(h^2-\tau^2)}{2h^2}\} \\
K_h^{1,2}(x,y;\tau) = R_1(x(y-\tau)_+) = x(y-\tau)_+ - \{\frac{y\,x}{h-x}(h-x-\tau)_+ \\
\qquad + (x+y-h)(y-\tau)_+ + (h-x-y)\frac{y}{h}(h-\tau)_+\} , \\
= (y-\tau)_+(h-y) - \frac{x\,y}{h-x}(h-x-\tau)_+ - \frac{y}{h}(h-x-y)(h-\tau) , \\
K_h^{2,1}(x,y;t,\tau) = R_1((x-t)_+\,(y-\tau)_+^o) = (x-t)_+[(y-\tau)_+^o - \frac{y}{h-x}(h-x-\tau)_+^o] .
\end{cases}
$$

Since the proof is a little bit lengthy we want to scetch what we are going to do: (11) implies that there are three nontrivial summands in (10), which we discuss separately. We first show how to find sharp error bounds and then treat the kernels $K_h^{2,1}$, $K_h^{o,3}$ and $K_h^{1,2}$.

Sharp error bounds: To find sharp error bounds we have to look at

$$(R_1 f)(x,y) = (R^{o,3}f)(x,y)+(R^{1,2}f)(x,y)+(R^{2,1}f)(x,y)$$

with $(R^{i,j}f)(x,y)$ defined corresponding to $K_h^{i,j}$ in (10) and (11). Let for example

$$T_+ := T_+^{2,1} := T_{+,(x,y)}^{2,1} := \{(t,\tau) \in T_h \mid K_h^{2,1}(x,y;t,\tau) \geqq o\}$$

$$T_- := T_-^{2,1} := T_{-,(x,y)}^{2,1} := \{(t,\tau) \in T_h \mid K_h^{2,1}(x,y;t,\tau) < o\} .$$

Then

$$|(R^{2,1}f)(x,y)| \leqq \iint_{T_-} + \iint_{T_-} |K_h^{2,1}(x,y;t,\tau)f^{(2,1)}(t,\tau)| \; dt\, d\tau$$

$$\leqq \|f^{2,1}\|_{L_\infty(T_h)} \cdot \iint_{T_h} | \; K_h^{2,1}(x,y;t,\tau) \; | \; dt\, d\tau$$

or

$$(12) \; \|R^{2,1}f\|_{L_\infty(T_h)} \leqq \|f^{2,1}\|_{L_\infty(T_h)} \cdot \| \iint_{T_h} | \; K_h^{2,1}(\cdot,\cdot;t,\tau)|dt\, d\tau\|_{L_\infty(T_h)} .$$

Since $f \in B_{1,2}^{\infty,3}$ the error bound (12) is sharp. To estimate $\|R^{2,1}f\|_{L_2(T_h)}$ we look at

$$\left((R^{2,1}f)(x,y)\right)^2 = \left(\iint_{T_h} K_h^{2,1}(x,y;t,\tau)f^{(2,1)}(t,\tau)dt\, d\tau\right)^2$$

$$\leqq \iint_{T_h} (K_h^{2,1}(x,y;t,\tau))^2 dt d\tau \cdot \|f^{(2,1)}\|^2_{L_2(T_h)}$$

or

$$\leqq \left(\iint_{T_h} K_h^{2,1}(x,y;t,\tau)dtd\tau\right)^2 \|f^{(2,1)}\|^2_{L_\infty(T_h)} .$$

Here the first inequality is sharp for every fixed $(x,y) \in T_h$, the second for all $(x,y) \in T_h$. So we finally have

$$
(13)\quad \begin{cases} \|R^{2,1}f\|_{L_2(T_h)} \leq \\[1ex] \leq \left(\iint_{T_h}\left(\iint_{T_h} K_h^{2,1}(x,y;t,\tau)dtd\tau\right)^2 dxdy\right)^{1/2} \cdot \|f^{(2,1)}\|_{L_\infty(T_h)} \\ \text{or} \\ \leq \iint_{T_h}\left(\iint_{T_h} (K_h^{2,1}(x,y;t,\tau))^2 dtd\tau\right) dxdy^{\;1/2} \cdot \|f^{(2,1)}\|_{L_2(T_h)} \; . \end{cases}
$$

In the first line the constant is best possible, whereas it is not in the second line.

Similar arguments hold for $R^{o,3}f$ and $R^{1,2}f$.

Now for $o \leq x$, $o \leq y$, $o \leq 1-x-y$, $o \leq \tau \leq 1$

$$K_h^{o,3}(\tfrac{x}{h},\tfrac{y}{h};\tfrac{\tau}{h}) = h^{-2}\, K_1^{o,3}(x,y;\tau) \quad \text{and } K_1^{o,3} = K_h^{o,3}|_{h=1} \;,$$

$$K_h^{1,2}(\tfrac{x}{h},\tfrac{y}{h};\tfrac{\tau}{h}) = h^{-2}\, K_1^{1,2}(x,y;\tau) \quad \text{and } K_1^{1,2} = K_h^{1,2}|_{h=1} \;,$$

$$K_h^{2,1}(\tfrac{x}{h},\tfrac{y}{h};\tfrac{t}{h},\tfrac{\tau}{h}) = h^{-1}\, K_1^{2,1}(x,y;t,\tau) \quad \text{and } K_1^{2,1} = K_h^{2,1}|_{h=1} \,.$$

So, for example, by (12)

$$\|R^{2,1}f\|_{L_\infty(T_h)} \leq \|f^{(2,1)}\|_{L_\infty(T_h)} \cdot \left\|\iint_{T_h} |\; K_h^{2,1}(\cdot,\cdot;t,\tau)\;|\; dtd\tau\right\|_{L_\infty(T_h)}$$

and with

$$T_1 := \{(x,y) \in \mathbb{R}^2 \mid o \leq x,\ o \leq y,\ o \leq 1-x-y\} = T_h|_{h=1} \;,$$

$$x' := \frac{x}{h} \;;\; y' := \frac{y}{h} \;,\; t' := \frac{t}{h} \;,\; \tau' := \frac{\tau}{h}$$

we have

$$\iint_{T_h} |\; K_h^{2,1}(x,y;t,\tau)\;|\; dtd\tau =$$

$$= \iint_{T_1} h^1 \;|\; K_1^{2,1}(x',y';t',\tau')\;|\; hdt'hd\tau'$$

$$= h^3 \iint_{T_1} |K_1^{2,1}(x',y';t',\tau')| dt'd\tau' \;.$$

In a similar way we have

$$\|R^{o,3}f\|^2_{L_2[o,h]} \leqq \iint_{T_h} \left(\iint_{T_h} (K^{o,3}_h(x,y;\tau))^2 d\tau\right) dxdy \, \|f^{o,3}(o,\cdot)\|^2_{L_2[o,h]}$$

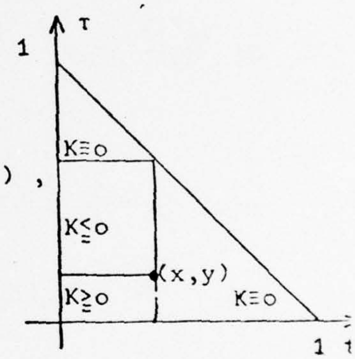
and

$$\iint_{T_h}\left(\iint_{T_h}(K^{o,3}_h(x,y;\tau))^2 d\tau\right)dxdy =$$

$$h^4 \iint_{T_1}\left(\iint_{T_1}(K^{o,3}_1(x',y';\tau'))^2 h d\tau'\right)hdx'hdy'$$

$$= h^7 \iint_{T_1}\left(\iint_{T_1}(K^{o,3}_1(x',y';\tau'))^2 d\tau'\right)dx'dy' \; .$$

So, to get the constants in (8) and (9), it is enough, to confine ourselves to $h = 1$ in $K^{o,3}_h$, $K^{1,2}_h$ and $K^{2,1}_h$.



ad $K^{2,1}$: From (11) we have

$$K^{2,1}_1(x,y;t,\tau) = (x-t)_+ \left((y-\tau)^o_+ - \frac{y}{1-x}(1-x-\tau)^o_+\right) ,$$

and therefore

$$(14) \quad \begin{cases} K^{2,1}_1 \equiv o \text{ for } x \leqq t \, , \\ K^{2,1}_1 = \dfrac{(x-t)(1-x-y)}{1-x} \geqq o \text{ for } o \leqq t \leqq x,\ o \leqq \tau \leqq y \, , \\ K^{2,1}_1 = -\dfrac{y(x-t)}{1-x} \leqq o \text{ for } o \leqq t \leqq x,\ y \leqq \tau \leqq 1-x \, , \\ K^{2,1}_1 \equiv o \text{ for } o \leqq t \leqq x,\ 1-x \leqq \tau \leqq 1 \, . \end{cases}$$

So

$$\iint_{T_1}|K^{2,1}_1(x,y;t,\tau)|dtd\tau = \int_{t=o}^{x}\left(\int_{\tau=o}^{y}\frac{1-x-y}{1-x}(x-t)d\tau + \int_{\tau=y}^{1-x}\frac{y}{1-x}(x-t)d\tau\right)dt$$

$$= -\frac{1-x-y}{1-x}\, y \,(x-t)^2\Big|_{t=o}^{x} = \frac{x^2y(1-x-y)}{1-x}$$

and therefore

$$|R^{2,1}f(x,y)| \leqq \|f^{(2,1)}\|_{L_\infty(T_1)} \frac{x^2y(1-x-y)}{1-x} \text{ for } (x,y) \in T_1 .$$

To find $\|g^{2,1}\|_{L_\infty(T_1)}$ with

(15) $g^{2,1}(x,y) := \frac{x^2y(1-x-y)}{1-x} \geqq o$ for $(x,y) \in T_1$, $= o$ iff $(x,y) \in \partial T_1$,

we derive:

$$\frac{\partial g^{2,1}}{\partial y} = \frac{x^2}{1-x}(1-x-2y) = o \quad \text{iff } x = o \quad \text{or} \quad 1-x = 2y$$

$$\frac{\partial g^{2,1}}{\partial x} = \frac{xy}{(1-x)^2}[2(1-x)^2 + y(x-2)] = o .$$

So $\frac{\partial g^{2,1}}{\partial x}(x,y) = \frac{\partial g^{2,1}}{\partial y}(x,y) = o$ iff $x = o$ or $8y^2-2y^2-y = o$,

$1-x = 2y$,

s.t. $y = \frac{1}{6}$, $x = \frac{2}{3}$ is the maximum point for $g^{2,1}$, so

$$\|g^{2,1}\|_{L_\infty(T_1)} = g^{2,1}(\tfrac{2}{3},\tfrac{1}{6}) = \tfrac{1}{27} \text{ and}$$

(16) $$\|R^{2,1}f\|_{L_\infty(T_h)} \leqq \frac{h^3}{27}\|f^{(2,1)}\|_{L_\infty(T_1)} .$$

To find estimations for $\|R^{2,1}f\|_{L_2(T_1)}$ we proceed as follows (see (14))

$$[(R^{2,1}f)(x,y)]^2 = \Big(\int_{t=o}^{x}\int_{\tau=o}^{1-x} K_1^{2,1}(x,y;t,\tau)f^{(2,1)}(t,\tau)dtd\tau\Big)^2$$

$$\leqq \int_{t=o}^{x}\int_{\tau=o}^{1-x}[K_1^{2,1}(x,y;t,\tau)]^2 dtd\tau \int_{t=o}^{x}\int_{\tau=o}^{1-x}[f^{(2,1)}(t,\tau)]^2 dtd\tau$$

$$\leqq \int_{t=o}^{x}\Big(\int_{\tau=o}^{y}\Big(\frac{1-x-y}{1-x}\Big)^2(x-t)^2 d\tau + \int_{\tau=y}^{1-x}\Big(\frac{y}{1-x}\Big)^2(x-t)^2 d\tau\Big)dt\ \|f^{(2,1)}\|^2_{L_2(T_1)}$$

$$= \frac{x^3y(1-x-y)}{3(1-x)}\|f^{(2,1)}\|^2_{L_2(T_1)} .$$

So finally $\|R^{2,1}f\|^2_{L_2(T_1)} \leq$

$$\leq \iint_{T_1} \frac{x^3 y(1-x-y)}{3(1-x)} dxdy \|f^{(2,1)}\|^2_{L_2(T_1)} = \frac{1}{18.60} \|f^{(2,1)}\|^2_{L_2(T_1)}$$

or, with $g^{2,1}$ from (15) and $\iint_{T_1} (g^{2,1}(x,y))^2 dxdy = 1/400.21$

$$(17) \quad \|R^{2,1}f\|_{L_2(T_h)} \leq \begin{cases} \dfrac{h^3}{6\sqrt{30}} \|f^{(2,1)}\|_{L_2(T_h)} \\[2ex] \dfrac{h^4}{20\sqrt{21}} \|f^{(2,1)}\|_{L_\infty(T_h)} . \end{cases}$$

ad $K^{0,3}$: From (11) we have for $h = 1$

$$(18) \quad K_1^{0,3}(x,y;\tau) = \frac{(y-\tau)_+^2}{2} - \frac{y}{2(1-x)} (1-x-\tau)_+^2 + \frac{y(1-x-y)(1-\tau^2)}{2} .$$

We first prove that

$$(19) \quad K_1^{0,3}(x,y;\tau) \geq 0 \quad \text{for } (x,y) \in T_1 \text{ and } \tau \in [0,1] .$$

In $0 \leq \tau \leq y$ we have

$$K_1^{0,3}(x,y;\tau) = \frac{(y-\tau)^2}{2} - \frac{y}{2(1-x)} (1-x-\tau)^2 + \frac{y(1-x-y)(1-\tau^2)}{2} .$$

Now

$$\frac{\partial K_1^{0,3}(x,y;\tau)}{\partial\tau} \text{ is linear in } \tau$$

$$\frac{\partial K_1^{0,3}(x,y;\tau)}{\partial\tau}\Big|_{\tau=0} \equiv 0$$

$$\frac{\partial K_1^{0,3}(x,y;\tau)}{\partial\tau}\Big|_{\tau=y} = y(1-x-y)\left(\frac{1}{1-x} - y\right) \geq 0 \text{ in } T_1$$

and $= 0$ iff $y=0$ or $1-x-y = 0$

$$K^{0,3}(x,y;\tau)\big|_{\tau=0} = 0.$$

This implies $K^{0,3}(x,y;\tau) \geq 0$ in $0 \leq \tau \leq y$.

For $o \leqq y \leqq \tau \leqq 1 - x$ we have (see (18))

$$K_1^{o,3}(x,y;\tau) = -\frac{y}{2(1-x)}\ [(1-x-\tau)^2 - (1-x-\tau+\tau-y)(1-\tau^2)(1-x)]$$

$$= +\frac{y}{2(1-x)}\ [(1-x-\tau)(1-\tau+x\tau)\tau + (\tau-y)(1-\tau^2)(1-x)] \geqq o\ .$$

For $o \leqq y \leqq 1 - x \leqq \tau \leqq 1$ $K_1^{o,3}$ reduces to

$$K_1^{o,3}(x,y;\tau) = \frac{y(1-x-y)(1-\tau^2)}{2} \geqq o,$$

so (19) is proved.

Since $K^{o,3}$ is a semidefinite kernel we have with $\eta \in (o,1)$

$$(R^{o,3}f)(x,y) = \int_o^h K_1^{o,3}(x,y;\tau)f^{(o,3)}(o,\tau)d\tau$$

$$= f^{(o,3)}(o,\eta)\int_o^h K_1^{o,3}(x,y;\tau)d\tau$$

and, analogously to (12) and (13),

$$\|R^{o,3}f\|_{L_\infty(T_h)} \leqq \|f^{(o,3)}(o,\cdot)\|_{L_\infty[o,h]}\ \left\| \int_o^h K_h^{o,3}(\cdot,\cdot;\tau)d\tau \right\|_{L_\infty(T_h)}$$

and

$$\|R^{o,3}f\|_{L_2(T_h)} \leqq \|f^{(o,3)}(o,\cdot)\|_{L_2[o,h]}\ \left\| \sqrt{\int_o^h (K_h^{o,3}(\cdot,\cdot;\tau))^2 d\tau} \right\|_{L_2(T_h)}\ .$$

Since we are interested only in the constants, it is enough to discuss $h=1$ in $K_h^{o,3}$. Corresponding to

$$K_1^{o,3}(x,y;\tau) = (\phi_1 + \phi_2 + \phi_3)(x,y;\tau) \text{ with}$$

$$\phi_1(x,y;\tau) := \frac{(y-\tau)_+^2}{2} = o \quad \text{for } y \leqq \tau \leqq 1$$

$$\phi_2(x,y;\tau) := -\frac{y}{2(1-x)}(1-x-\tau)_+^2 = o \text{ for } 1 - x \leqq \tau \leqq 1$$

$$\phi_3(x,y;\tau) := \frac{y(1-x-y)(1-\tau^2)}{2}$$

we find

$$G^{o,3}(x,y) := \int_o^1 (K_1^{o,3}(x,y;\tau))^2 d\tau = \int_o^y (\phi_1^2 + 2\phi_1\phi_2 + 2\phi_1\phi_3)d\tau$$

$$+ \int_y^{1-x} (\phi_2^2 + 2\phi_2\phi_3)d\tau + \int_{1-x}^1 \phi_3^2 \, d\tau$$

$$= \left[(\frac{1}{2o} + \frac{1}{12}) \, y^5 - \frac{y^4(1-x)}{6} - \frac{y^6}{6o(1-x)} + \frac{y^4(1-x-y)}{6} (1 - \frac{y^2}{1o}) \right]$$

$$+ \left[\frac{y^2(1-x)^3}{2o} - \frac{y^2(1-x-y)(1-x)^2}{6} (1 - \frac{(1-x)^2}{1o}) \right] + \frac{2y^2(1-x-y)^2}{15}$$

and finally

$$(20) \quad \begin{cases} \iint_{T_1} G^{o,3}(x,y)dxdy = \dfrac{397}{31752oo} \\ \\ \sqrt{\dfrac{397}{31752oo}} = \dfrac{\sqrt{397/2}}{126o} \end{cases}$$

Computing

$$\tilde{G}^{o,3}(x,y) := \int_o^1 K_1^{o,3}(x,y;\tau)d\tau = \int_o^y (\phi_1 + \phi_2 + \phi_3)d\tau$$

$$+ \int_y^{1-x} (\phi_2 + \phi_3)d\tau + \int_{1-x}^1 \phi_3 \, d\tau$$

we find

$$\tilde{G}^{o,3}(x,y) = \frac{y}{6}(y^2 - (1-x)^2 + 2(1-x-y)) = \frac{y}{6}[(y-1)^2 - x^2] \geqq o$$

since $1-x-y \geqq o$ and therefore $(y-1)^2 \geqq x^2$. Because of

$$\frac{\partial \tilde{G}^{o,3}(x,y)}{\partial x} = \frac{y}{6}\,(+\ 2(1-x)-2) = -\frac{yx}{3} = o \text{ iff } x = o \text{ or } y = o$$

the extrema of $\tilde{G}^{o,3}(\cdot,\cdot)$ in T_1 are realized on ∂T_1:

$$o \leqq \tilde{G}^{o,3}(x{=}o,y) = \frac{y(y-1)^2}{6} \leqq \frac{2}{81}\ ,$$

$$\tilde{G}^{o,3}(x,y{=}o) \equiv o\ ,$$

$$\tilde{G}^{o,3}(x{=}1-y,y) \equiv o\ .$$

So

(21) $\quad \|\tilde{G}^{o,3}(\cdot,\cdot)\|_{L_\infty(T_1)} = \frac{2}{81}$.

ad $K^{1,2}$: With $h = 1$ (11) implies

(22) $\quad K_1^{1,2}(x,y;\tau) = (y-\tau)_+\,(1-y) - \frac{xy}{1-x}\,(1-x-\tau)_+ - (1-x-y)\,y(1-\tau)$.

Since $K_1^{1,2}$ is a continuous *piecewise linear* function (for $x \neq 1$), one finds

(23) $\quad K_1^{1,2}(x,y;\tau) \leqq o$

by simply checking $K^{1,2}(x,y\,;\,\tau)$ for $\tau = o,\ y,\ 1-x,\ 1$.

Again we have to compute

$$G^{1,2}(x,y) := \int_o^1 (K_1^{1,2}(x,y;\tau))^2\,d\tau \quad \text{and}$$

$$\tilde{G}^{1,2}(x,y) := -\int_o^1 K_1^{1,2}(x,y;\tau)\,d\tau \quad \text{(see (23))}$$

and find

$$G^{1,2}(x,y) = \frac{1}{3}\Big[(1-y)y^3(-2+3y-xy-y^2 + \frac{xy}{1-x}) + xy^2(1-x)(2-2y-x^2-xy) + (1-x-y)^2y^2\Big]$$

(24) $\quad \iint_{T_1} G^{1,2}(x,y)\,dxdy = \left(\frac{\sqrt{19/14}}{2o}\right)^2$,

$$-\tilde{G}^{1,2}(x,y) = \frac{y}{2}(x^2-y^2+2y-1) = \frac{y}{2}(x^2-(1-y)^2) \leqq 0 \ .$$

Because of $\partial\tilde{G}^{1,2}/\partial x = 0$ iff $x = 0$ or $y \doteq 0$ we find $\|\tilde{G}^{1,2}\|_{L_\infty(T_1)}$ by discussing

$$0 \leqq \tilde{G}^{1,2}(x=0,y) = +\frac{y(1-y)^2}{2} \leqq +\frac{2}{27}$$

$$\tilde{G}^{1,2}(x,y=0) \equiv 0$$

$$\tilde{G}^{1,2}(x=1-y,y) = 0$$

or

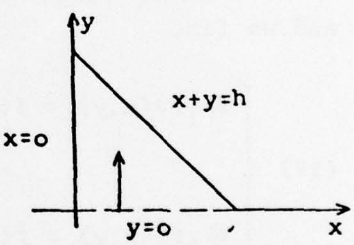
(25) $$\|\tilde{G}^{1,2}\|_{L_\infty(T_1)} = \frac{2}{27} \ .$$

By fitting together formulas (16), (21) and (25) we have (8), by (17), (20), (24) we have (9) and the constants are best possible in the sense indicated in Theorem 2 and in the following Remark. □

4. Second Interpolation Scheme:

By interchanging x and y we come from Problem 1 to

Problem 2: *For* $f : T_h \to \mathbb{R}$ *with* $f^{(0,1)}$ *existing on* $y=0$ *find a blending function* G_2f *such that*



(26) $$\begin{cases} G_2f|_{x=0} = f|_{x=0} & \textit{for } y \in [0,h] \\ G_2f|_{x+y=h} = f|_{x+y=h} & \textit{for } x \in [0,h] \\ (G_2f)^{(0,1)}|_{y=0} = f^{(0,1)}|_{y=0} & \textit{for } x \in [0,h] \ . \end{cases}$$

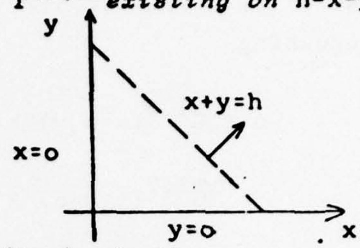
The corresponding results to Theorem 1 and 2 are valid , too, and we have for G_2 :

(27) $$\begin{cases} (G_2f)(x,y) = \dfrac{h-x-y}{h-y} f(0,y) + \dfrac{x}{h-y} f(h-y,y) + (x+y-h)f^{(0,1)}(x,0) \\ \quad + (h-x-y)\left[\dfrac{x}{h^2}(f(h,0) - f(0,0)) + \dfrac{h-x}{h} f^{(0,1)}(0,0) + \dfrac{x}{h}\left(f^{(0,1)}(h,0) + f^{(1,0)}(h,0)\right)\right. \end{cases}$$

5. Third Interpolation Scheme and Remainder:

Now we discuss

Problem 3: *For* $f : T_h \to \mathbb{R}$ *with* $f^{(o,1)} + f^{(1,o)}$ *existing on* $h-x-y = o$ *find a blending function* $G_3 f$ *with*



$$(28)\quad \begin{cases} G_3 f|_{x=o} = f|_{x=o} & \text{for } x \in [o,h] \\ G_3 f|_{y=o} = f|_{y=o} & \text{for } y \in [o,h] \\ (G_3 f^{(1,o)} + G_3 f^{(o,1)})|_{h-x-y=o} = (f^{(1,o)} + f^{(o,1)})|_{h-x-y=o} . & \end{cases}$$

By distinguishing the two different cases $x \geqq y$ and $x \leqq y$ we define $G_1 f$ and $G_2 f$ by

$$Q_1 f|_{y=o} = f|_{y=o},$$

$$(Q_1 f^{(1,o)} + Q_1 f^{(o,1)})|_{h-x-y=o} = (f^{(1,o)} + f^{(o,1)})|_{h-x-y=o},$$

$$Q_1 f|_{y=o} = f|_{y=o},$$

$$(Q_2 f^{(1,o)} + Q_2 f^{(o,1)})|_{h-x-y=o} = (f^{(1,o)} + f^{(o,1)})|_{h-x-y=o},$$

and we find

$$(29)\quad \begin{cases} (Q_1 f)(x,y) = f(x-y,o) + y(f^{(1,o)} + f^{(o,1)})(\frac{h+x-y}{2}, \frac{h-x+y}{2}) , \\ \\ (Q_2 f)(x,y) = f(o,y-x) + x(f^{(1,o)} + f^{(o,1)})(\frac{h+x-y}{2}, \frac{h-x+y}{2}) \end{cases}$$

and finally

$$(30)\quad (G_3 f)(x,y) := \begin{cases} Q_1 f(x,y) \text{ for } y \leqq x , \\ \\ Q_2 f(x,y) \text{ for } y \geqq x . \end{cases}$$

A straightforward computation yields

Theorem 3: *Let* $f : T_h \to \mathbb{R}$ *be given and* $\|f(\cdot,h-\cdot)\|_{L_\infty[o,h]} = \operatorname{supess}_{x\in[o,h]} |f(x,h-x)|$.

(a) If $(f^{(1,o)} + f^{(o,1)})$ *exists along* h-x-y=o, *then* G_3f *is a solution of* Problem 3. *Moreover, if* $f(\cdot,o)$, $f(o,\cdot)$ *and* $(f^{(1,o)} + f^{(o,1)})(\cdot,h-\cdot) \in C^j[o,h]$, *then* $G_3f \in C^j(T_h)$.

(b) G_3 *reproduces linear functions.*

(c) If $f(\cdot,o)$, $f(o,\cdot)$, $(f^{(1,o)} + f^{(o,1)})(\cdot,h-\cdot) \in L_\infty[o,h]$ *then*

$$\|G_3f\|_{L_\infty(T_h)} \leq \|f(\cdot,o)\|_{L_\infty[o,h]} + \|f(o,\cdot)\|_{L_\infty[o,h]} + $$

$$+ h\|(f^{(1,o)} + f^{(o,1)})(\cdot,h-\cdot)\|_{L_\infty[o,h]} .$$

To discuss the error we, again, introduce

$$R_3f := f - G_3f . \tag{31}$$

Since we have to transform the original triangle T_h, we have to transform the conditions defining $B^{r,m}_{p,q}(a,b)$, too. Let

$$\hat{B}^{r,m}_{p,q} := \left\{ \begin{array}{l} f : T_h \to \mathbb{R} \mid o \leq i , j : \\ \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^i \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^j f \in \left\{ \begin{array}{l} C(T_h) \text{ for } (i,j) < (p,q) , \\ AC(T_h \{x=y\}) \text{ for } j<q,\ i=m-j-1, \\ L_r(T_h \{x=y\}) \text{ for } j<q,\ i=m-j, \\ AC(T_h \{x+y=h\}) \text{ for } i<p,\ j=m-i-1, \\ L_r(T_h \{x+y=h\}) \text{ for } i<p,\ j=m-i. \\ L_r(T_h) \text{ for } (i,j) = (p,q) \end{array} \right. \end{array} \right\} \tag{32}$$

Theorem 4: *If* $f \in \hat{B}^{\infty,2}_{2,1}$ *then, with* $\|f(\cdot,h-\cdot)\|_{L_\infty[o,h]} = \operatorname{supess}_{x\in[o,h]} |f(x,h-x)|$,

$$\left\{ \begin{array}{l} \|R_3f\|_{L_\infty(T_h)} \leq \frac{h^2}{8}\left\|\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f(\cdot,h-\cdot)\right\|_{L_\infty[o,h]} \\ \qquad + \frac{h^3}{108}\left\|\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) f\right\|_{L_\infty(T_h)} . \end{array} \right. \tag{33}$$

If $f \in \hat{B}^{2,2}_{2,1}$ *then*

$$(34)\quad \begin{cases} \|R_3 f\|_{L_2(T_h)} \leq \dfrac{h^{5/2}}{6\sqrt{10}\sqrt{2}} \|(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2 f(\cdot, h-\cdot)\|_{L_2[o,h]} \\ \qquad + \dfrac{h^3}{48\sqrt{5}} \|(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2 (\frac{\partial}{\partial y} - \frac{\partial}{\partial x}) f\|_{L_2(T_h)} . \end{cases}$$

The constants given in (33) *and* (34) *are best possible and a similar remark to* Remark 1 *is possible.*

<u>Proof:</u> To be able to apply (4) and (5) we transform the variables:

$$(35)\quad u := \frac{x+y}{\sqrt{2}}, \; v := \frac{y-x}{\sqrt{2}}, \; x = \frac{u-v}{\sqrt{2}}, \; y = \frac{u+v}{\sqrt{2}} .$$

Transforming $Q_2 f$ via (35) we find with (29), (30) for $y \geq x$ or $v \geq o$

$$(\tilde{Q}_2 f)(u,v) = g(v,v) + (u-v) g^{(1,o)}(\frac{h}{\sqrt{2}}, v)$$

with

$$(36)\quad \begin{cases} g(u,v) := f(x = \frac{u-v}{\sqrt{2}}, y = \frac{u+v}{\sqrt{2}}), \\ (\tilde{Q}_2 f)(u=v,v) = g(v,v) = f(o, \sqrt{2} v) \text{ for } v \in [o, \frac{h}{\sqrt{2}}] , \\ \frac{\partial}{\partial u} (\tilde{Q}_2 f)(u = \frac{h}{\sqrt{2}}, v) = g^{(1,o)}(\frac{h}{\sqrt{2}}, v) = \\ \qquad = \frac{1}{\sqrt{2}} (f^{(1,o)} + f^{(o,1)})(x=h-y, y) . \end{cases}$$

Applying (4) and (5) to

$$(\tilde{R}_3 \tilde{Q}_2 f)(u,v) := (g - \tilde{Q}_2 f)(u,v)$$
$$= g(u,v) - g(v,v) - (u-v)\, g^{(1,o)}(\frac{h}{\sqrt{2}}, v)$$

with $m = 2 = p$, $q = 1$, $a = \frac{h}{\sqrt{2}}$ and $b = o$ we have

$$(37)\quad \begin{cases} (\tilde{R}_3\tilde{Q}_2 f)(u,v) = \int\limits_0^{h/\sqrt{2}} K_h^{2,o}(u,v;t)\, g^{(2,o)}(t,o)dt \\[2ex] + \int\limits_0^{h/\sqrt{2}} K_h^{o,2}(u,v;\tau) g^{(o,2)}(\frac{h}{\sqrt{2}},\tau)d\tau + \int\limits_0^{h/\sqrt{2}} K_h^{1,1}(u,v;\tau)g^{(1,1)}(\frac{h}{\sqrt{2}},\tau)d\tau \\[2ex] + \int\limits_{\tau=o}^{h/\sqrt{2}} \int\limits_{t=\tau}^{h/\sqrt{2}} K_h^{2,1}(u,v;t,\tau)g^{2,1}(t,\tau)dtd\tau\ . \end{cases}$$

By (6)

$$K_h^{2,o}(u,v;t) = \tilde{R}_3((u-t)\psi(\frac{h}{\sqrt{2}},t,u))$$

and with

$$\psi(\frac{h}{\sqrt{2}},t,u) = \begin{cases} -1 \text{ for } o \leqq u \leqq t \leqq h/\sqrt{2} \\ o \quad \text{for } o \leqq t < u \leqq h/\sqrt{2} \end{cases} = -(t-u)_+^{\,o}$$

$$K_h^{2,o}(u,v;t) = \tilde{R}_3((t-u)_+) = (t-u)_+ - (t-v)_+$$

$$K_h^{o,2}(u,v;\tau) = \tilde{R}_3((v-\tau)_+) \equiv o$$

$$K_h^{1,1}(u,v;\tau) = \tilde{R}_3((u-\frac{h}{\sqrt{2}})(v-\tau)_+^{\,o}) \equiv o$$

$$K_h^{2,1}(u,v;t,\tau) = \tilde{R}_3((u-t)\psi(\frac{h}{\sqrt{2}},t,u) \cdot (v-\tau)^o\psi(o,\tau,v))$$

$$= \tilde{R}_3((t-u)_+(v-\tau)_+^{\,o}) = ((t-u)_+ - (t-v)_+)(v-\tau)_+^{\,o}\ .$$

Similar arguments like in the proof of Theorem 1 show that (here $T_h^* := \{(u,v) \mid o \leqq v \leqq u \leqq \frac{h}{\sqrt{2}}\}$, $T_1^* := T_h^*|_{h=1}$)

$$(38)\quad \begin{cases} \| \int\limits_0^{h/\sqrt{2}} K_h^{2,o}(\cdot,\cdot;t)g^{(2,o)}(t,o)dt \|_{L_\infty(T_h^*)} \\[3ex] \leqq h^2 \| \int\limits_0^{1/\sqrt{2}} K_1^{2,o}(\cdot,\cdot,t)dt \|_{L_\infty(T_1^*)} \cdot \|g^{(2,o)}(\cdot,o)\|_{L_\infty[o,\frac{h}{\sqrt{2}}]} \end{cases}$$

$$(39)\quad \begin{cases} \| \int_0^{h/\sqrt{2}} K_h^{2,o}(\cdot,\cdot;t)g^{(2,o)}(t,o)dt \|_{L_\infty(T_h^*)} \\ \leq h^{5/2} \left\| \sqrt{\int_0^{1/\sqrt{2}} (K_1^{2,o}(\cdot,\cdot;t))^2 dt} \right\|_{L_2(T_1^*)} \cdot \|g^{(2,o)}(\cdot,o)\|_{L_2[o,\frac{h}{\sqrt{2}}]}, \end{cases}$$

$$(40)\quad \begin{cases} \| \iint_{T_h^*} K_h^{2,1}(\cdot,\cdot;t,\tau)g^{(2,1)}(t,\tau)dtd\tau \|_{L_\infty(T_h^*)} \\ \leq h^3 \| \iint_{T_1^*} K_1^{2,1}(\cdot,\cdot;t,\tau)dtd\tau \|_{L_\infty(T_1^*)} \cdot \|g^{(2,1)}\|_{L_\infty(T_h^*)} \end{cases}$$

and so on. Now

$$K_1^{2,o}(u,v;t) = \begin{cases} -(u-v) \leq o & \text{for } o \leq v \leq u \leq t \leq 1/\sqrt{2} \\ -(t-v) \leq o & \text{for } o \leq v \leq t < u \leq 1/\sqrt{2} \\ o & \text{for } o \leq t \leq v \leq u \end{cases}$$

and so

$$\int_0^{1/\sqrt{2}} K_1^{2,o}(u,v;t)dt = -(u-v)\left(\frac{1}{\sqrt{2}} - \frac{u+v}{2}\right)$$

$$\int_0^{1/\sqrt{2}} (K_1^{2,o}(u,v;t))^2 dt = (u-v)^2\left(\frac{1}{\sqrt{2}} - \frac{2u+v}{3}\right) .$$

That implies

$$(41)\quad \| \int_0^{1/\sqrt{2}} K_1^{2,o}(\cdot,\cdot;t)dt \|_{L_\infty(T_1^*)} = \frac{1}{4} ,$$

$$(42)\quad \| \int_0^{1/\sqrt{2}} (K_1^{2,o}(\cdot,\cdot;t))^2 dt \|_{L_2(T_1^*)} = \frac{1}{3\sqrt{10\sqrt{2}}} .$$

For the second remainder term we need

$$K_1^{2,1}(u,v;t,\tau) = \begin{cases} o & \text{for } v \leqq \tau, \\ -(u-v) \leqq o & \text{for } v > \tau \text{ and } o \leqq v \leqq u \leqq t \leqq 1/\sqrt{2}, \\ -(t-v) \leqq o & \text{for } v > \tau \text{ and } o \leqq v \leqq t < u \leqq 1/\sqrt{2}, \\ o & \text{for } v > \tau \text{ and } o < t \leqq v \leqq u \leqq 1/\sqrt{2}. \end{cases}$$

Therefore

$$\iint_{T_1^*} K_1^{2,1}(u,v;t,\tau)dtd\tau = -(u-v)\ v\ (\frac{1}{\sqrt{2}} - \frac{u+v}{2}) \ ,$$

$$\iint_{T_1^*} (K_1^{2,1}(u,v;t,\tau))^2 dtd\tau = (u-v)^2\ v\ (\frac{1}{\sqrt{2}} - \frac{2u+v}{3})$$

and finally

$$(43)\quad \| \iint_{T_1^*} (K_1^{2,1}(\cdot,\cdot;t,\tau))dtd\tau \|_{L_\infty(T_1^*)} = \frac{\sqrt{2}}{54}$$

$$(44)\quad \| \sqrt{\iint_{T_1^*} (K_1^{2,1}(\cdot,\cdot;t,\tau))^2 dtd\tau} \|_{L_2(T_1^*)} = \frac{1}{12\sqrt{10}} \ .$$

Until now we have only treated the error in T_h^*, not in

$$T_{h,-}^* := \{(u,v) \mid o \leqq -v \leqq u \leqq h/\sqrt{2}\} \ .$$

Because of the symmetry of $Q_1 f$ and $Q_2 f$ in (29) we find, for example,

$$(45)\begin{cases} \left\| \int_{-h/\sqrt{2}}^{h/\sqrt{2}} K_h^{2,o}(\cdot,\cdot;t)g^{(2,o)}(t,o)dt \right\|_{L_\infty[-h/\sqrt{2},h/\sqrt{2}]} \\ \leqq h^2 \left\| \int_o^{1/\sqrt{2}} K_1^{2,o}(\cdot,\cdot;t)dt \right\|_{L_\infty(T_1^{::})} \\ \qquad \cdot \max\{ \|g^{(2,o)}(\cdot,o)\|_{L_\infty[o,h/\sqrt{2}]}, \|g^{(2,o)}(\cdot,o)\|_{L_\infty[-h/\sqrt{2},o]}\} \\ = h^2 \left\| \int_o^{1/\sqrt{2}} K_1^{2,o}(\cdot,\cdot;t)dt \right\|_{L_\infty(T_1^{::})} \cdot \|g^{(2,o)}(\cdot,o)\|_{L_\infty[-h/\sqrt{2},h/\sqrt{2}]} \end{cases}$$

and

$$(46)\begin{cases} \left\| \iint_{T_h^{::} \cup T_{h,-}^{::}} K_h^{2,1}(\cdot,\cdot;t,\tau)g^{(2,1)}(t,\tau)dtd\tau \right\|_{L_2(T_h^{::} \cup T_{h,-}^{::})} \\ \leqq h^3 \left\| \sqrt{\iint_{T_1^{::}} (K_1^{2,1}(\cdot,\cdot;t,\tau))^2 dtd\tau} \right\|_{L_2(T_1^{::})} \\ \qquad \cdot \left[\|g^{(2,1)}\|_{L_2(T_h^{::})} + \|g^{(2,1)}\|_{L_2(T_{h,-}^{::})} \right] \\ = h^3 \left\| \sqrt{\iint_{T_1^{::}} (K_1^{2,1}(\cdot,\cdot;t,\tau))^2 dtd\tau} \right\|_{L_2(T_1^{::})} \|g^{(2,1)}\|_{L_2(T_h^{::} \cup T_{h,-}^{::})}. \end{cases}$$

Finally we have to use the inverse transformation to (35).
By (35)

$$\frac{\partial}{\partial u} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial v} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right), \quad v = o \text{ iff } x = y,$$

and so, see for instance (39), (40),

$$
(47) \quad \begin{cases} g^{(2,o)}(u,o) = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f(x,h-x)\ , \\ \\ g^{(2,1)}(u,v) = \frac{1}{2\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) f(x,y)\ . \end{cases}
$$

Fitting together (38), (41), (43), (45), (47) we find (33), whereas (39), (42), (44), (46), (47) gives (34). □

If one imposes additional symmetry conditions on f, one can construct interpolating functions satisfying (28) and reproducing bivariate polynomials.

Acknowledgements: This report was partially supported by a grant of the Volkswagen-Stiftung for the first author and a Fullbright Fellowship for the second author. Both authors had the chance to stay at the Mathematics Research Center in Madison during this time. We want to thank the sponsors and Dipl.-Math. B. Schmitt for double-checking the lengthy but elementary computations and Mrs. M. Zahn who typed the manuscript.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: (14) MRC-TSR - 1744	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: (9) Technical	
4. TITLE (and Subtitle): (6) BLENDING INTERPOLATION SCHEMES ON TRIANGLES WITH ERROR BOUNDS.	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): K. Böhmer and Gh. Coman (10) Klaus/Boehmer George/Coman	8. CONTRACT OR GRANT NUMBER(s): (15) DAAG29-75-C-0024 ~~112256~~
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 6 (Spline Functions and Approximation Theory)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.	12. REPORT DATE: (11) April 1977
	13. NUMBER OF PAGES: 24 (12) 27p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Surface interpolation
Blending interpolation on triangles
Interpolation error

This document presents

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~We give~~ rational approximation schemes interpolating the values of a given function along two edges of a triangle and the normal derivative along the third edge. In addition, ~~we give~~ are given error bounds for ~~our~~ these schemes. For the uniform norm, these bounds are best possible.

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